Monte Carlo Techniques

Monte Carlo:

The use of statistical methods to solve math problems that may or may not (initially) involve probability

Monte Carlo Algorithm:

1) Devise a random variable whose mean is the solution
2) Devise a way to generate samples
3) Collect statistics

Why Use Monte Carlo?

Advantages:

1) Usually easy to formulate (independent of problem)
2) Scales well (easy in any dimension)

Disadvantages:

1) Solutions are imprecise or “noisy”
2) Can be very slow

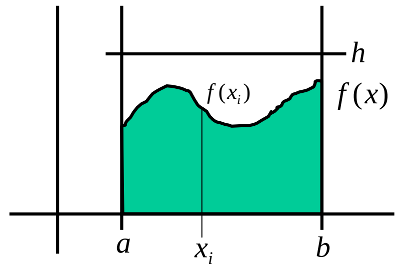
Basic Monte Carlo

Sample with uniformly distributed random points

$$\alpha = \int_a^b f(x)\, dx$$

x_i is a uniformly distributed random variable on $[a,b]$

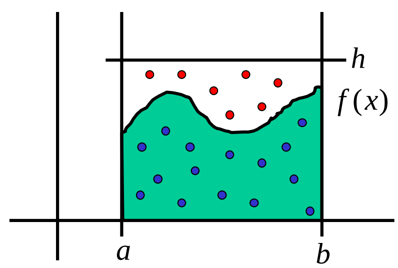
$$\alpha \approx \frac{b-a}{N} \sum_{i=1}^{N} f(x_i)$$



Rejection Method

Compute the integral as a percent of an area

$$\alpha = \int_a^b f(x)\, dx$$
$$= \lim_{N \to \infty} \frac{N^*}{N} h(b-a)$$



N^* is the # of samples below $f(x)$

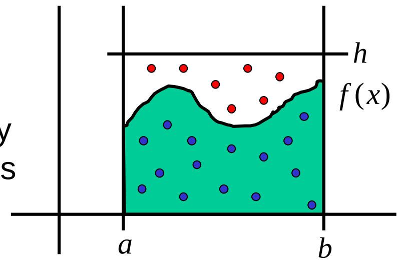
N is the total # of samples

Rejection Method

Compute the integral as a percent of an area

$$\alpha \approx \frac{N^*}{N} h(b-a)$$

Sample points are uniformly distributed random variables on $[a,b]\times[0,h]$



$$\xi \sim [0,1]$$
$$(x,y) = (a+(b-a)\xi_1, h\xi_2)$$
$$\text{if } y \le f(x) \text{ then } N^{*}{+}{+}$$

Why is Monte Carlo Good Here?

Why not use quadrature?

1) Integrand may be discontinuous

2) Quadrature is useless in high dimensions

Probability Density Function

Probability Density Function (PDF):

is a function from [a,b] to *R* such that:

1) $f(x) \geq 0 \qquad \forall x \in [a,b]$

2) $\int_a^b f(x)\, dx = 1$

3) $P(x_1 \leq x < x_2) = \int_{x_1}^{x_2} f(x)\, dx$

Theorem

If g is a PDF on $[a,b]$, then

$$\int_a^b h(x) g(x)\, dx = \int_a^b h(x) \left(g(x)\, dx\right)$$

$$= E[h(X)]$$

where X is distributed (~) according to g

Primary Estimators

Using a uniform PDF

$$\alpha = \int_a^b f(x)\,dx$$

$$= \int_a^b [f(x)(b-a)]\frac{1}{(b-a)}\,dx$$

$$= E[f(X)(b-a)] \qquad X \sim \frac{1}{b-a}$$

$$\approx f(x_i)(b-a)$$

where x_i is drawn from X

One sample is usually not enough

Average Estimator

Take an average of multiple samples

$$\alpha \approx \frac{b-a}{N}\sum_{i=1}^{N} f(x_i)$$

where x_i are drawn from X

This is what we called basic monte carlo

Why can't we use a nonuniform PDF?

NonUniform PDFs

We can rewrite any integral with a PDF

$$\alpha = \int_a^b f(x)\,dx$$

$$= \int_a^b \frac{f(x)}{g(x)} \left(g(x)\,dx\right)$$

$$\approx \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{g(x_i)} \qquad x_i \sim \frac{1}{b-a}$$

But what is the best PDF to choose?

Variance Reduction

Variance $\sigma^2(Z) = \langle Z^2 \rangle - \langle Z \rangle^2$

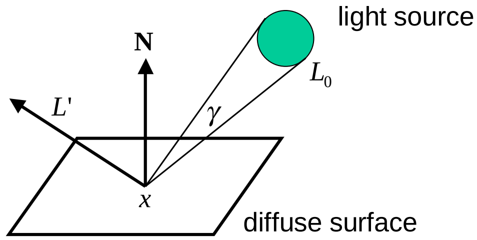
If $Z = \dfrac{f(X)}{g(X)}$ with $X \sim g$

then $\sigma^2(Z) = \displaystyle\int_a^b \frac{f^2(x)}{g(x)} dx \quad - \quad \alpha^2$

$g(x) = \dfrac{f(x)}{\int_a^b f(x)\,dx}$ will give zero variance

This is known as importance sampling

Monte Carlo in Image Synthesis



$$L' = \frac{\rho}{\pi} \int_{\Omega} L(\omega) \cos\theta \, d\omega$$

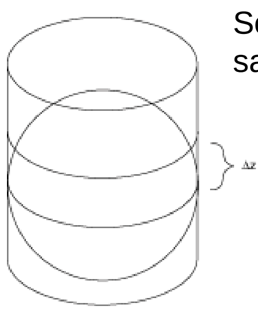
$$= \frac{\rho \, L_0}{\pi} \int_{\gamma} \cos\theta \, d\omega$$

How to sample the source?

Uniform Sampling

Archimedes Theorem

Given a sphere inscribed within a cylinder, for any Δz, both surfaces have equal area



So we can generate uniform samples by sampling the cylinder and projecting inwards

$$z = 1 - 2\xi_1$$

$$\theta = 2\pi\xi_2$$

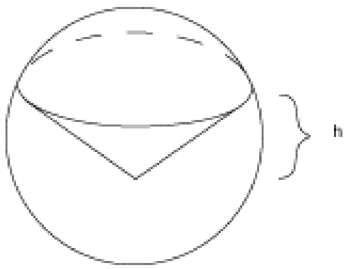
$$x = \cos\theta \sqrt{1 - z^2}$$

$$y = \sin\theta \sqrt{1 - z^2}$$

Uniform Sampling

Sampling the top of the sphere

Sample the cylinder and project inwards



$$z = 1 - (1-h)\xi_1$$

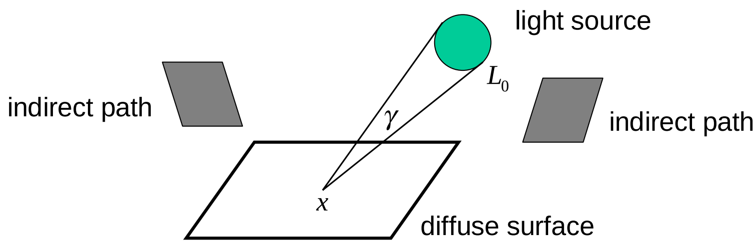
$$\theta = 2\pi\xi_2$$

$$x = \cos\theta\sqrt{1-z^2}$$

$$y = \sin\theta\sqrt{1-z^2}$$

This is useful for sampling the light sources

Incorporating Indirect Lighting



$$L' = \frac{\rho}{\pi}\int_{\Omega} L(\omega)\cos\theta\, d\omega$$

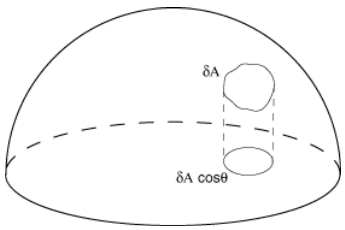
$$= \frac{\rho}{\pi}\int_{\Omega-\gamma} L(\omega)\cos\theta\, d\omega + \frac{\rho L_0}{\pi}\int_{\gamma}\cos\theta\, d\omega$$

This is known as stratification

Weighted Sampling

Sampling the hemisphere with a $\cos\theta$ distribution

Sample the base and project upwards



$x, y =$ uniform sampling of the disk

$$z = \sqrt{1 - x^2 - y^2}$$

But how do we sample the disk?

Weighted Sampling

Sampling the disk

If $\xi \in [0,1]$ is uniformly distributed, then $X = F^{-1}(\xi)$, where F is the CDF of g, is distributed according to g

Proof

$$\boxed{\begin{aligned} P(X \le x) &= P(F^{-1}(\xi) \le x) \\ &= P(\xi \le F(x)) \\ &= F(x) \end{aligned}}$$

Disk sampling is thus:

$$r = \sqrt{\xi_1}$$

$$\theta = 2\pi\xi_2$$

$$x = r\cos\theta$$

$$x = r\sin\theta$$

Monte Carlo Rendering